On the algebraic solutions of the sixth Painlevé equation related to second order Picard-Fuchs equations

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Abstract

We describe two algebraic solutions of the sixth Painlevé equation which are related to (isomonodromic) deformations of Picard-Fuchs equations of order two.

1 Statement of the result

In this note we describe two algebraic solutions of the following Painlevé VI ($\mathcal{P}_{VI}$) equation

$$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \frac{1}{2}(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t})(\frac{d\lambda}{dt})^2-(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t})\frac{d\lambda}{dt}\\ &\quad+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t^2-1)}[\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}].\end{aligned}$$

related to deformations of Picard-Fuchs equations of special type. Recall that the $\mathcal{P}_{VI}$ equation governs the isomonodromic deformations of the second

order Fuchsian equations

$$x'' + p_1(s)x' + p_2(s)x = 0, \quad ' = \frac{d}{ds} \tag{1}$$

with 5 singular points, one of which is apparent [7]. Suppose that the solution of the Fuchsian equation (1) is given by an Abelian integral

$$x(s) = \int_{\gamma_s} \omega$$

where ω is a rational one-form on $\mathbb{C}^2$, $\Gamma_s \subset \mathbb{C}^2$ is a family of algebraic curves depending rationally on s, and $\gamma_s \subset \Gamma_s$ is a continuous family of closed loops. Then the equation (1) is said to be of Picard-Fuchs type and its monodromy group is conjugated to a subgroup of $\mathbf{Gl}_2(\overline{\mathbb{Q}})$ (generically $\mathbf{Gl}_2(\mathbb{Z})$). For this reason any continuous deformation

$$a \to \Gamma_{s,a}$$

of the family Γ_s induces an isomonodromic deformation of (1). If in addition $\Gamma_{s,a}$ depends algebraically in a, the coefficients of (1) are also algebraic functions in a, and hence they provide an algebraic solution of $\mathcal{P}_{VI}$.

From now on we denote

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, \frac{1}{2} - \delta).$$

Our main result is the following

Theorem 1 *The pencil of $\mathcal{P}_{VI}(\alpha)$ equations*

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{1}{8}, \frac{s}{8}, \frac{s}{8}, \frac{s}{8}), s \in \mathbb{C} \tag{2}$$

has a common algebraic solution parameterized as

$$\lambda = \frac{a^2(2-a)}{a^2 - a + 1}, t = \frac{a^3(2-a)}{2a-1}, a \in \mathbb{C}. \tag{3}$$

The $\mathcal{P}_{VI}(\alpha)$ equation with

$$\alpha = (\frac{1}{8}, \frac{1}{2}, 0, 0)$$

has an algebraic solution parameterized as

$$\lambda = \frac{a(a-2)(2a^2 + a + 2)}{a^2 - 7a + 1}, t = \frac{a^3(2-a)}{2a-1}, a \in \mathbb{C} \tag{4}$$

The meaning of these solutions is the following. Consider the family of elliptic curves

$$\Gamma_s = \{(\xi, \eta) \in \mathbb{C}^2 : \eta^2 + \frac{3}{2a-1}\xi^4 - \frac{4(a+1)}{2a-1}\xi^3 + \frac{6a}{2a-1}\xi^2 = s\} \tag{5}$$

and let $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ be a family of cycles depending continuously on $s \in \mathbb{C}$. The Abelian integral of first kind

$$\int_{\gamma(s)} \frac{d\xi}{\eta}$$

satisfies a Picard-Fuchs equation of second order depending on a parameter a, defining an isomonodromy deformation of the equation. This deformation corresponds then to an algebraic solution of $\mathcal{P}_{VI}(\alpha)$ given by (3). In a similar way, the Abelian integral of second kind

$$\int_{\gamma(s)} \frac{(3\xi^2 - 2(a+1))\xi d\xi}{\eta}$$

satisfies a Picard-Fuchs equation of second order. The isomonodromy deformation of this equation with respect to a is described by the solution (4) of $\mathcal{P}_{VI}$ equation.

Algebraic solutions of $\mathcal{P}_{VI}$ were found by many authors, e.g. Hitchin[6], Manin[8], Dubrovin-Mazzocco[4], Boalch[3]. Dubrovin and Mazzocco classified all algebraic solutions of the $\mathcal{P}_{VI}$ equation corresponding to

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{1}{2}(2\mu - 1)^2, 0, 0, 0), \mu \in \mathbb{R}.$$

It turns out that these solutions, up to symmetries, are in a one-to-one correspondence with the regular polyhedra in the three dimensional space. Our solution (3) with

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1/8, 0, 0, 0)$$

corresponds then to the tetrahedron solution of Dubrovin-Mazzocco ($\mu = +1/4$). It is identified to their solution (A_3) via the Okamoto type transformation (1.24),(1.25), see [4]. It is remarkable that the same solution, but for

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1/8, 1/8, 1/8, 1/8)$$

was also found by Hitchin[6]. This shows that (3) is a common solution to the family (2) of $\mathcal{P}_{VI}$ equations. It is clear that for transcendental values of

s in (2) the corresponding isomonodromic family of Fuchs equations (1) can not be of Picard-Fuchs type.

The paper is organized as follows. In the next section we recall briefly, following [7], the relationship between $\mathcal{P}_{VI}$ and the isomonodromic deformations of Fuchs equations. In section 3 we deduce the relevant Picard-Fuchs equation and establish the main result.

The present text is an abridged version of [2].

2 The Garnier system and the $\mathcal{P}_{VI}$ equation.

Consider a Fuchsian differential equation

$$x'' + p_1(s)x' + p_2(s)x = 0, \quad ' = \frac{d}{ds} \tag{1}$$

with *five singular points, exactly one of which is apparent*. After a bi-rational change of the independent variable s and a linear change of the dependent variable x (involving s) we may suppose that the singular points are $0, 1, t, \lambda, \infty$, where the singularity λ is apparent and the corresponding Riemann scheme is

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ 0 & 0 & 0 & 0 & \alpha \\ \theta_1 & \theta_2 & \theta_3 & k & \alpha+\theta_\infty \end{pmatrix}, n \in \mathbb{N}, 2\alpha + \sum_i \theta_i + n = 3.$$

In what follows we shall always suppose that $n = 2$ (which is satisfied generically)).

The coefficients p_1, p_2 are easily computed to be

$$p_1(s) = \frac{1-\theta_1}{s} + \frac{1-\theta_2}{s-1} + \frac{1-\theta_3}{s-t} - \frac{1}{t-\lambda}$$

$$p_2(s) = \frac{k}{s(s-1)} - \frac{t(t-1)K}{s(s-1)(s-t)} + \frac{\lambda(\lambda-1)\mu}{s(s-1)(s-\lambda)}$$

where μ is a constant

$$k = \frac{1}{4}\{(\sum_{i=1}^{3} \theta_i - 1)^2 - \theta_\infty^2\}.$$

The compatibility condition for the singular point λ to be apparent reads

$$
\begin{aligned}
K = K(\lambda,\mu,t) &= \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_2(\lambda-1)(\lambda-t) \\
&\quad +\theta_3\lambda(\lambda-t) + (\theta_1-1)\lambda(\lambda-1)\}\mu + k\lambda].
\end{aligned}
$$

From the discussion above it is seen that the Fuchs equation (1) depends on the parameters $\theta_0,\theta_1,\theta_t,\theta_\infty,\lambda,\mu,t$. Let us denote this equation by $E_\theta(\lambda,\mu,t)$.

Theorem 2 $\lambda(t),\mu(t)$ *is a solution of the Garnier system*

$$
\begin{aligned}
\frac{d\lambda}{dt} &= \frac{\partial K}{\partial \mu} \\
\frac{d\mu}{dt} &= -\frac{\partial K}{\partial \lambda}.
\end{aligned}
$$

if and only if the induced deformation of $E_\theta(\lambda,\mu,t)$ *is isomonodromic.*

It is straightforward to check that the sixth Painlevé system $\mathcal{P}_{VI}(\alpha)$ with parameters

$$
\alpha = (\frac{1}{2}\theta_\infty^2, \frac{1}{2}\theta_0^2, \frac{1}{2}\theta_1^2, \frac{1}{2}\theta_t^2) \tag{6}
$$

is equivalent to the Garnier system. We get therefore the following
Corollary. If

$$
(t,\lambda,\mu) \to (t,\lambda(t),\mu(t))
$$

is an isomonodromic deformation of $E_\theta(\lambda,\mu,t)$, then $\lambda(t)$ is a solution of $\mathcal{P}_{VI}(\alpha)$ equations with parameters given by (6).

3 Picard-Fuchs equations

In this section we restrict our attention to the deformation

$$
f_a(\xi,\eta) = \eta^2 + \frac{3}{2a-1}\xi^4 - \frac{4(a+1)}{2a-1}\xi^3 + \frac{6a}{2a-1}\xi^2, a \in \mathbb{C}
$$

of the singularity $\eta^2+\xi^4$ of type A_3, see [1]. The critical values of $f_a(\xi,\eta)$ are

$$
0, 1, t = \frac{a^3(2-a)}{2a-1}.
$$

Consider the locally trivial smooth fibration

$$f^{-1}(\mathbb{C} \setminus \{0, 1, t\}) \to \mathbb{C} \setminus \{0, 1, t\}$$

whose fibers the affine curves Γ_s, (5), $s \in \mathbb{C}\setminus\{0, 1, t\}$. Each Γ_s is topologically a torus with two removed points. Hence $\dim H_1(\Gamma_s, \mathbb{Z}) = \dim H^1_{DR}(\Gamma_s, \mathbb{C}) = 3$. Therefore if $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ is a family of cycles depending continuously on s, then the Abelian integral

$$I(s) = \int_{\gamma(s)} \omega, \omega = P(\xi, \eta)d\xi + Q(\xi, \eta)d\eta, P, Q \in \mathbb{C}[\xi, \eta]$$

satisfies a Fuchsian differential equation of order three, whose coefficients are polynomials in s, a. In the case when the differential form ω has no residues, it satisfies a second order equation. Explicitly, if $\gamma_1(s)$, $\gamma_2(s)$, is a continuous family of cycles generating the homology group of the compactified elliptic curve Γ_s, then the equation reads

$$\det \begin{pmatrix} x & x' & x'' \\ \int_{\gamma_1(s)} \omega & (\int_{\gamma_1(s)} \omega)' & (\int_{\gamma_1(s)} \omega)'' \\ \int_{\gamma_2(s)} \omega & (\int_{\gamma_2(s)} \omega)' & (\int_{\gamma_2(s)} \omega)'' \end{pmatrix} = 0.$$

It follows from the Picard-Lefschetz formula and the moderate growth of the integrals, that the coefficients of the above differential equations are rational in s, a. A local analysis of the singularities shows for instance that

$$\det \begin{pmatrix} \int_{\gamma_1(s)} \omega & (\int_{\gamma_1(s)} \omega)' \\ \int_{\gamma_2(s)} \omega & (\int_{\gamma_2(s)} \omega)' \end{pmatrix} = \frac{p(s, a)}{s(s-1)(s-t)}$$

where $p(s, a)$ is a polynomial in s, a. If we put $\omega = dx/y$ then $\int_{\gamma_1(s)} \omega$ grows no faster than $s^{1/4-1/2}$ at ∞ (for a fixed a). Thus

$$\frac{p(s, a)}{s(s-1)(s-t)}$$

grows at infinity no faster than $s^{-1/2-1}$ and hence no faster than s^{-2}. It is expected therefore than $p(s, a)$ is of degree one in s and the corresponding root, which we denote by λ, is an apparent singularity for the Picard-Fuchs equation in consideration. We are therefore in a position to apply Theorem 2, provided that the deformation of the Fuchs equation with respect to

the parameter a is isomonodromical. Indeed, the monodromy group of our equation is contained in $SL(2,\mathbb{Z})$ which shows that any deformation of this equation is isomonodromical. The Picard-Lefschetz formula shows that the monodromy group in question is generated, up to conjugacy, by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7}$$

To deduce an explicit formula for the corresponding algebraic solution of $\mathcal{P}_{VI}$ we need explicit formulae for the Picard-Fuchs equations.

Lemma 1 *Let $\gamma(s) \in H_1(\Gamma_s,\mathbb{Z})$ be a family of cycles depending continuously on s. The complete elliptic integrals of first and second kind*

$$x(s) = \int_{\gamma(s)} \frac{d\xi}{\eta}, \quad y(s) = \int_{\gamma(s)} \frac{(3\xi^2 - 2(a+1))\xi d\xi}{\eta}$$

satisfy Picard-Fuchs equations of the form

$$a_0(s)x'' + a_1(s)x' + a_2(s)x = 0$$
$$b_0(s)y'' + b_1(s)y' + b_2(s)y = 0$$

where

$$\begin{aligned}
a_0(s) &= s(s-1)((2a-1)s + a^3(a-2))((a^2-a+1)s + a^2(a-2)) \\
a_1(s) &= 2(2a-1)(a^2-a+1)s^3 + (a^6 - 3a^5 + 9a^4 - 19a^3 + 9a^2 - 3a + 1)s^2 \\
&\quad +2a^2(a-2)(a^4 - 2a^3 - 2a + 1)s - a^5(a-2)^2 \\
a_2(s) &= (2a-1)[27(a^2-a+1)s^2 - (a-2)(2a^4 - a^3 - 60a^2 - a + 2)s \\
&\quad +a^2(a-2)^2(10a^2 + 11a + 10)]/144 \\
b_0(s) &= s(s-1)((2a-1)s + a^3(a-2))((a^2-7a+1)s - a(a-2)(2a^2+a+2)) \\
b_1(s) &= (2a-1)s[(a^2-7a+1)s^2 - 2a(a-2)(2a^2+a+2)s \\
&\quad -a(a-2)^2(a^4+a^3+a^2+a+1)] \\
b_2(s) &= -(2a-1)[9(a^2-7a+1)s^2 - (a-2)(10a^4 + 31a^3 - 12a^2 + 31a + 10)s \\
&\quad -a(a-2)^2(2a^2+a+2)^2]/144
\end{aligned}$$

The proof of the above Lemma is straightforward, see for instance [5]. It is seen that the roots of $a_0(s)$ are $0, 1$ and

$$\lambda = \frac{a^2(2-a)}{a^2-a+1}, t = \frac{a^3(2-a)}{2a-1}$$

which implies the algebraic solution (3). In the same way the roots of $b_0(s)$ provide the solution (4). The Riemann schemes of the Picard-Fuchs equations for $x(s), y(s)$ are given by

$$\left(\begin{array}{ccccc} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a^2(2-a)}{a^2-a+1} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 2 & \frac{3}{4} \end{array}\right)$$

and

$$\left(\begin{array}{ccccc} 0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a(a-2)(2a^2+a+2)}{a^2-7a+1} & \infty \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 2 & -\frac{1}{4} \end{array}\right).$$

The Corollary after Theorem 2 implies that the curve (3) is an integral curve of the $\mathcal{P}_{VI}(\alpha)$ equation with parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{1}{8}, 0, 0, 0)$, see (6). Similarly, the Fuchsian equation satisfied by the complete elliptic integral of second kind $y(s)$ provides the algebraic solution (4) of $\mathcal{P}_{VI}(\alpha)$ with $\alpha = (\frac{1}{8}, \frac{1}{2}, 0, 0)$.

It is remarkable that (3) was found to be a solution of $\mathcal{P}_{VI}(\alpha)$ by Hitchin [6][p.177], but for $\alpha = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. After taking the difference between these two equations we obtain the following affine equation of the integral curve (3)

$$-\frac{t}{\lambda^2} + \frac{t-1}{(\lambda-1)^2} - \frac{t(t-1)}{(\lambda-t)^2} = 0.$$

This also shows that (3) is a common algebraic solution of the pencil of $\mathcal{P}_{VI}(\alpha)$ equations

$$\alpha = (\frac{1}{8}, \frac{s}{8}, \frac{s}{8}, \frac{s}{8}), s \in \mathbb{C}.$$

This completes the proof of Theorem 1.

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